1. A: We can solve this part by keeping track of the amount of money that Justin has over the whole ordeal. Justin first gets a fake \$100 from Shreyas so he has no money. He then gets \$100 from Prabhas after exchanging the counterfeit for real money. By giving adequate change to Shreyas he loses \$70, leaving him with \$30. Finally Prabhas takes \$100 back from Justin leaving him with -\$70 which means Justin lost $\underline{\$70}$.

   B: The ideal weighing method is by dividing the 9 pencils into 3 groups of 3 pencils. We now weight 2 of those groups together. If the scale is heavier on one side, we know that that group has Dylan's pencil. If the scale is balanced, then the leftover group has Dylan's pencil. We now take two random pencils from our group of 3, and with a similar logic we can deduce which is the heavier pencil. This means it only takes a total of $\underline{2}$ weight comparisons

   Final Answer: $70 + 2 = \boxed{72}$

2. Let's figure out the range of each function and then see where the functions overlap to find out when two of the following statements are true.

   A: Since $\cos(2x) = \cos^2(x) - \sin^2(x)$, we can simplify our expression to $\cos^2(x) + \sin^2(x) > \sin(x) + 1$ and then $0 > \sin(x)$. This gives us the range of $\underline{(\pi, 2\pi)}$.

   B: By multiplying by $\sin(x)$ on both sides of the inequality we get that $\tan(x) > 1$ when $\sin(x)$ is positive but $\tan(x) < 1$ when $\sin(x)$ is negative. This gives us the range $\underline{(\frac{\pi}{4}, \frac{\pi}{2}) \cup (\pi, \frac{5\pi}{4}) \cup (\frac{3\pi}{2}, 2\pi)}$

   C: One way you can do this problem is by simplifying this inequality into $2\sin(x)\cos(x) - 2\cos(x) > \sin(x) - 1$ by using sine double angle identities. We now can divide our inequality by $\sin(x) - 1$, and since $\sin(x) - 1$ is always negative, we must switch the sign. This gives us $\cos(x) < \frac{1}{2}$ which gives us a range of $\underline{(\frac{\pi}{3}, \frac{5\pi}{3})}$

   Final Answer: The range where only two of these ranges intersect is $(\frac{\pi}{3}, \frac{\pi}{2}) \cup (\frac{5\pi}{4}, \frac{3\pi}{2}), \cup (\frac{5\pi}{3}, 2\pi)$. This length of this range is $\frac{3\pi}{4}$ out of the total 2π which is equivalent to a probability of $\boxed{\frac{3}{8}}$

3. X: If we subtract the 2 equations and then isolate t, we get $\frac{x-y}{2} = t$. If we sub this back into our first equation and rearrange, we get $x^2 - 2xy + y^2 - 2x - 2y + 4 = 0$. The discriminant of this conic is 0, which means this conic is a parabola, which means it has an eccentricity of $\underline{1}$.

   Y: If we divide by $\sqrt{3}$ on both sides in our first equation, we can then square both of our equations and then add them to get $\frac{x^2}{3} + y^2 = \frac{1 - 2t^2 + t^4 + 4t^2}{1 + 2t^2 + t^4}$ which simplifies to $\frac{x^2}{3} + y^2 = 1$. The is simply an ellipse that has an eccentricity of $\underline{\frac{\sqrt{6}}{3}}$

   Final Answer: $1 + \frac{\sqrt{6}}{3} = \boxed{\frac{3+\sqrt{6}}{3}}$

4. A: The geometric mean of the arithmetic mean and the harmonic mean of a and b is equal to $\sqrt{(\frac{a+b}{2})(\frac{2ab}{a+b})} = \sqrt{ab}$, and since 529 is 23^2 and 729 is 27^2, then the answer is simply $23 \cdot 27$ or $\underline{621}$

   B: By Splitting the $\frac{1}{x}$ term, we get $x^2 + \frac{1}{2x} + \frac{1}{2x}$. Now we can use the AM-GM inequality on the 3 terms in our equation to find that $(\frac{1}{3})(x^2 + \frac{1}{x}) \geq \sqrt[3]{\frac{1}{4}}$. This means that the minimum value of the function $x^2 + \frac{1}{x}$ will be $\underline{3 \cdot 4^{-1/3}}$

   Final Answer: $621 \cdot \log_4 \left(4^{-1/3}\right) = \boxed{-207}$

5. A: If the dot product of 2 vectors is equal to their cross product, we can deduce that $|v||w|\sin\theta = |v||w|\cos\theta$. This means that $\sin\theta = \cos\theta$ and that the angle between the vectors is $45°$. We can now use law of cosines with

the 2 vectors to find the distance between the end points of the vectors. This gives us $A^2 = (4\sqrt{2})^2 + (7\sqrt{2})^2 - 2(4\sqrt{2})(7\sqrt{2})(\cos(45°))$. This means that for this part is $A^2 = \underline{130 - 56\sqrt{2}}$

B: Since we have deduced in the last part that the angle between the 2 vectors is 45°, we can now simply use the SAS formula for area of a triangle to find that the area is $\frac{1}{2}(4\sqrt{2})(7\sqrt{2})(\sin(45°))$ or $\underline{14\sqrt{2}}$

C: The formula for the projection of a vector v onto a vector w is $\frac{v \cdot w}{|w|^2}w$ This means that the length of the projection vector will be $\frac{(4\sqrt{2})(7\sqrt{2})\cos(45°)}{98}7\sqrt{2} = \underline{4}$

Final Answer: $130 - 56\sqrt{2} + 4(14\sqrt{2}) + 4 = \boxed{134}$

6. A: We can simply look at the last digits of powers of 7 to find out which cases are divisible by 5. There are 4 possible last digits for each term which means there is 16 possible cases. The only cases where the sum is divisible by 5 is if the last digits are (1,9), (9,1), (3,7), and (7,3). This means there are 4 out of the 16 cases that work or $\underline{\frac{1}{4}}$.

   B: $\tan(3285°) = \tan(3285° - 9(360°)) = \tan(3285° - 3240°) = \tan(45°) = \underline{1}$

   C: The two graphs are circles that have radius $\frac{1}{2}$ and are centered at $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. To find the area of the intersection between the 2 circles, we simply take the area of the sector defined by the center of one of the circles and the two intersection points between the two circles and then subtract the area of the triangle defined by those same 3 points. This gives us a lune of area $\frac{\pi}{16} - \frac{1}{8}$. Since there are 2 lunes making the intersection, we can simply double the value of our lune area to get our answer of $\underline{\frac{\pi}{8} - \frac{1}{4}}$

   Final Answer: $8B(A + C) = \boxed{\pi}$

7. Final Answer: This question is actually much simpler than it seems. Adding $9x + n$ to x simply results in $10x + n$, which is the same as adding n to the end so this increases the length by 1. Adding $10^{\lfloor \log(x)+1 \rfloor} \cdot n$ is the same as adding $n \cdot 10^{length(x)}$ which is the same as adding n to the front so this increases the length by 1. Adding n itself never changes the length at all (at least for the first week), so it's pretty much useless. This means that there is a 2/3 chance each time that the length will increase by 1, and a 1/3 chance that it will stay roughly the same. For x to be greater than 10^5 on Day 6, its length must be greater than or equal to 6. Since 6 digits can be added overall, x can only be 6 or 7 digits. Then we can simply use binomial theorem to find both probabilities: $\binom{6}{5} \cdot \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right) + \binom{6}{6} \cdot \left(\frac{2}{3}\right)^6 = \boxed{\frac{256}{729}}$.

8. Final Answer: If we take the first term of each part we realize that this forms a geometric series with first term and common ratio $\frac{1}{p+q}$. We can use sum of an infinite geometric series formula to realize that this series sums to $\frac{1}{p+q-1}$. This means that the final answer will be equal to $\sum_{p=1}^{2020} \sum_{q=1}^{2021-p} \frac{1}{p+q-1}$. This series is equal to $1 + \frac{1}{2} + \frac{1}{3} \dots \frac{1}{2020} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots \frac{1}{2020} + \frac{1}{3} + \frac{1}{4} \dots \frac{1}{2020} + \cdots$. This is also equivalent to $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{3}\right) + 4\left(\frac{1}{4}\right) + 5\left(\frac{1}{5}\right) \dots 2020\left(\frac{1}{2020}\right) = \boxed{2020}$

9. A: Note that $\lim_{x \to 0}(1 + kx)^{\frac{m}{x}} = e^{mk}$. Since $1 + \frac{4x}{(3x+1)^2} = \frac{(1+x)(1+9x)}{(1+3x)^2}$, then we can break up the limit into the product of $\lim_{x \to 0}(1 + x)^{\frac{2}{x}}$ and $\lim_{x \to 0}(1 + 9x)^{\frac{2}{x}}$, divided by the square of $\lim_{x \to 0}(1 + 3x)^{\frac{2}{x}}$. This means that A is $\frac{(e^2)(e^{18})}{(e^6)^2} = \underline{e^8}$

   B: The numerator of the fraction is $\left(2n \ln\left(1 + \frac{1}{n}\right)\right)^x = \left(\ln\left(\left(1 + \frac{1}{n}\right)^{2n}\right)\right)^x$, and by the Taylor Series of e, the summation is now $e^{\ln\left(\left(1+\frac{1}{n}\right)^{2n}\right)} = \left(1 + \frac{1}{n}\right)^{2n}$. Taking the limit to infinity gives $\underline{e^2}$

Final Answer: $\ln(e^8 \cdot e^2) = \boxed{10}$

10. A: For this part, our limit is $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-(x)^2}{h} = \lim_{h\to 0} \frac{2hx+h^2}{h} = \lim_{h\to 0} 2x+h = \underline{2x}$

    B: For this part our limit is $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = \underline{\frac{\sqrt{x}}{2x}}$

    C: For this part $f'(x) = \lim_{h\to 0} \frac{(x+h)^2-3(x+h)+4-(x^2-3x+4)}{h} = \lim_{h\to 0} \frac{2hx+h^2-3h}{h} = \lim_{h\to 0} 2x+h-3 = 2x-3$. $f''(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)-3-(2x-3)}{h} = 2$. Now we use these values to solve for our Mclaurin Series to get $\frac{4}{0!} + \frac{-3x}{1!} + \frac{2x^2}{2!} = \underline{x^2-3x+4}$.

    Final: $C + (AB)^2 = x^2 - 2x + 4$ so $p = 1, q = 2, r = 4$ and $p + q + 4 = \boxed{7}$.

11. A: Above the x-axis is a triangle with base 2 and height 1. Below the x-axis is a right triangle with legs of length 1. This means the integral evaluates to $\frac{1}{2} \cdot 2 \cdot 1 - \frac{1}{2} \cdot 1 \cdot 1 = \underline{\frac{1}{2}}$

    B: $y = \sqrt{9-4x^2}$ which simplifies to $y^2 + 4x^2 = 9$ which further simplifies to the ellipse $\frac{x^2}{\frac{9}{4}} + \frac{y^2}{9} = 1$. The area of this ellipse is $\frac{3}{2} \cdot 3 \cdot \pi = \frac{9\pi}{2}$. However the function is bounded by the lines $x = 0$ and $x = 1.5$ which cut the area of the ellipse in half. And once again the area of the ellipse is cut in half because the function isn't defined for negative values of y, since y can never be negative. This means the integral evaluates to $\underline{\frac{9\pi}{8}}$

    C: Simplifying the term in the integral, we get $\sin(2x) + 1$. Then after drawing our borders, we realize we have to find the area of two periods of this function. However, we can see that this area is exactly $\frac{1}{2}$ of the area of the rectangle bounded by the x-axis, $x = -\frac{\pi}{4}$, $x = \frac{7\pi}{4}$, and $y = 2$. The area of this rectangle is 4π, so half of that gives us an answer of $\underline{2\pi}$.

    Final Answer: $\frac{\frac{9\pi}{8}}{\frac{1}{2} \cdot 2\pi} = \boxed{\frac{9}{8}}$

12. Final Answer: First we should note that $\det(AB) = \det(A)\det(B)$. We should also notice that the 2nd matrix is simply transpose of the first matrix, but with every value multiplied by 2. The transpose of a matrix has the same determinant as the original matrix, and since the matrix has each of the 4 rows multiplied by 2, the determinant will be multiplied by $2 \cdot 2 \cdot 2 \cdot 2$ or 16 times the 1st matrix. So if we call the determinant of our first matrix D, the answer to the problem will be $4|D|$. Now we simply have to find D. If we subtract 3 times the first column from the 3rd column and then subtract 2 times the 2nd row from the 1st row, we get the matrix $\left|\begin{bmatrix} 0 & -7 & 0 & 0 \\ 4 & 5 & 0 & 5 \\ 2 & 4 & 2 & 9 \\ 1 & 12 & 0 & 2 \end{bmatrix}\right|$ which has a determinant equal to $2(7)(4(2) - 5(1)) = 42$. So, the final answer is $4 \cdot 42$ or $\boxed{168}$.

13. A: We should first note that triangular numbers are defined by $\frac{n(n+1)}{2}$. This means that part A is equivalent to $\frac{1}{\sum_{x=1}^{\infty} \frac{2}{n(n+1)}}$. Using Partial Fraction Decomposition on the denominator we can simplify our expression to $\frac{1}{\sum_{x=1}^{\infty} \frac{2}{n} - \frac{2}{n+1}} = \underline{\frac{1}{2}}$

B: The answer to this part is equivalent to $\frac{1}{\sum_{n=1}^{\infty}\frac{n}{3^n}}$. Now the denominator evaluates to $S=\frac{1}{3}+\frac{2}{9}+\frac{3}{27}\ldots$ and if we take $3S-S$ we get that $2S=1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\ldots=\frac{3}{2}$ which means that $S=\frac{3}{4}$ and that the answer to this part is $\underline{\frac{4}{3}}$

C: Testing out the first few reciprocal terms, we see they are $1,3,\frac{9}{2},\frac{27}{6}$. In fact, the sum is equivalent to $\sum_{n=0}^{\infty}\frac{3^n}{n!}$ (which can be shown by induction), which is equal to e^3. The reciprocal of this sum is $\underline{\frac{1}{e^3}}$.

Final Answer: $ABC=\left(\frac{1}{2}\right)\left(\frac{4}{3}\right)\left(\frac{1}{e^3}\right)=\boxed{\frac{2}{3e^3}}$.

14. F: The cube roots of unity are $e^0, e^{2\pi i/3}$ and $e^{4\pi i/3}$. $e^{2\pi i/3}$ is the only one that is in the 2nd quadrant so, the answer is $\underline{e^{2\pi i/3}}$.

A: The 5th roots of unity are $e^0, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$. The only one of these that lies in the 4th quadrant is $\underline{e^{8\pi i/5}}$

M: $cis(48^\circ)=cis\left(\frac{4\pi}{15}\right)=\underline{e^{4\pi i/15}}$

Final Answer: $\frac{FA}{M}=\frac{e^{2\pi i/3}\cdot e^{8\pi i/5}}{e^{4\pi i/15}}=e^{\frac{2\pi i}{3}+\frac{8\pi i}{5}-\frac{4\pi i}{15}}=e^{\frac{30\pi i}{15}}=e^{2\pi i}=\boxed{1}$

15. P: By Law of Sines, $\frac{\tan(A)}{\sin(A)}=\frac{\tan(B)}{\sin(B)}=\frac{\tan(C)}{\sin(C)}$, so $\sec(A)=\sec(B)=\sec(C)=2R$. This implies $A=B=C=60^\circ$, so the triangle is equilateral with side length $\tan(60^\circ)=\sqrt{3}$. Then the area of the triangle is $A=\frac{3\sqrt{3}}{4}$, the semiperimeter is $s=\frac{3\sqrt{3}}{2}$, and the product of the side lengths is $3\sqrt{3}$. The inradius is $\frac{A}{s}=\frac{1}{2}$ and the circumradius is $\frac{abc}{4A}=1$, so the bounded area is $\pi-\frac{\pi}{4}=\underline{\frac{3\pi}{4}}$.

Q: The circumradius of $\triangle DEF$ is equal to $\frac{A}{\sin(a)}=\frac{12}{\sin(60^\circ)}=\frac{12}{\frac{\sqrt{3}}{2}}=8\sqrt{3}$. Since the length of the latus rectum is 4 times the length of the distance between the vertex and the focus, we can maximize the length of the latus rectum by maximizing the distance between the vertex and the focus. To do this, we can put the vertex and the focus on opposite sides of the circle such that the distance between them is the diameter of the circumcircle. This means that the maximum length of the latus rectum is $4(2R)=8R=8(8\sqrt{3})=\underline{64\sqrt{3}}$

Final Answer: $\frac{1}{2}\left(\frac{3\pi}{4}\right)\left(64\sqrt{3}\right)(\sin 30^\circ)=\boxed{12\pi\sqrt{3}}$.